SUPERMATRIX REPRESENTATIONS OF SEMIGROUP BANDS

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Abstract

Various semigroups of noninvertible supermatrices of the special (antitriangle) shape having nilpotent Berezinian which appear in supersymmetric theories are defined and investigated. A subset of them continuously represents left and right zero semigroups and rectangular bands. The ideal properties of higher order rectangular band analogs and the "wreath" version of them are studied in detail. We introduce the "fine" equivalence relations leading to "multidimesional" eggbox diagrams. They are full images of Green's relations on corresponding subsemigroups.

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1 Introduction

Matrix semigroups [46, 50, 51] are the great tool in concrete and thorough investigation of detail abstract semigroup theory structure [11, 23, 29, 36]. Matrix representations [38, 49] are widely used in studying of finite semigroups [33, 52, 65], topological semigroups [3, 9] and free semigroups [8, 21]. Usually matrix semigroups are defined over a field $\mathbb{K}$ [40, 48, 47]. Nevertheless, after discovering of supersymmetry [61, 62] the realistic unified particle theories began to be considered in superspace [22, 56]. In this picture all variables and functions were defined not over a field $\mathbb{K}$, but over Grassmann-Banach superalgebras over $\mathbb{K}$ [53, 63] (or their generalizations [42, 43, 57]). However, the noninvertible (and therefore semigroup) character of them was ignored for a long time, and only recently the consistent studies of semigroups in supersymmetric theories appeared [15, 16, 17]. In addition to their physical contents these investigations led to some nontrivial pure mathematical constructions having unusual properties connected with noninvertibility and zero divisors [14, 18]. In particular, it was shown [19] that supermatrices of the special shape can form various strange and sandwich semigroups not known before [25, 37].

In this paper we work out continuous supermatrix representations of semigroup bands[1] introduced in [19]. The Green's relations on continuous zero semigroups and wreath rectangular bands are studied in detail. We introduce the "fine" equivalence relations which generalize them in some extent and lead to the "multidimensional" analog of eggbox diagrams. Next investigations are connected with continuous superanalogs of 0-simple semigroups and Rees's theorem which will appear elsewhere.

2 Preliminaries

Let Λ be a commutative Banach $\mathbb{Z}_2$-graded superalgebra over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$) with a decomposition into the direct sum: $\Lambda = \Lambda_0 \oplus \Lambda_1$. The elements a from Λ_0 and Λ_1 are homogeneous and have the fixed even and odd parity defined as $|a| \stackrel{def}{=} \{i \in \{0, 1\} = \mathbb{Z}_2 | \, a \in \Lambda_i\}$. If Λ admit the

[1] We note that study of idempotent semigroup representations [58], especially the matrix ones [20], is important by itself. The idempotents also appear and are widely used in random matrix semigroup applications [6, 12, 39].

decomposition into body and soul [53] as $\Lambda = \mathbb{B} \oplus \mathbb{S}$, where $\mathbb{B}$ and $\mathbb{S}$ are purely even and odd algebras over $\mathbb{K}$ respectively, the even homomorphism $\mathfrak{r}_{body} : \Lambda \to \mathbb{B}$ is called a body map and the odd homomorphism $\mathfrak{r}_{soul} : \Lambda \to \mathbb{S}$ is called a soul map [4, 44]. Usually Λ is modelled with the Grassmann algebras $\wedge(N)$ having N generators [53, 60] or $\wedge(\infty)$ [7, 10, 54]. The soul $\mathbb{S}$ is obviously a proper two-sided ideal of Λ which is generated by Λ_1. These facts allow us to consider noninvertible morphisms on a par with invertible ones (in some sense), which gives many interesting and nontrivial results (see e.g. [15, 16, 17]).

We consider $(p|q)$-dimensional linear model superspace $\Lambda^{p|q}$ over Λ (in the sense of [5, 34]) as the even sector of the direct product $\Lambda^{p|q} = \Lambda_0^p \times \Lambda_1^q$ [53, 60]. The even morphisms $\mathrm{Hom}_0\left(\Lambda^{p|q}, \Lambda^{m|n}\right)$ between superlinear spaces $\Lambda^{p|q} \to \Lambda^{m|n}$ are described by means of $(m+n) \times (p+q)$-supermatrices[2] (for details see [5, 35]).

3 Supermatrix semigroups

We consider $(1+1) \times (1+1)$-supermatrices describing the elements from $\mathrm{Hom}_0\left(\Lambda^{1|1}, \Lambda^{1|1}\right)$ in the standard $\Lambda^{1|1}$ basis [5]

$$M \equiv \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{Mat}_\Lambda(1|1) \tag{1}$$

where $a, b \in \Lambda_0$, $\alpha, \beta \in \Lambda_1$ (in the following we use Latin letters for elements from Λ_0 and Greek letters for ones from Λ_1, and all odd elements are nilpotent of index 2). For sets of matrices we also use corresponding bold symbols, e.g. $\mathbf{M} \stackrel{def}{=} \{M \in \mathrm{Mat}_\Lambda(1|1)\}$, and the set product is $\mathbf{M} \cdot \mathbf{N} \stackrel{def}{=} \{\cup MN \mid M, N \in \mathrm{Mat}_\Lambda(1|1)\}$.

In this $(1|1)$ case the supertrace defined as $\mathrm{str} : \mathrm{Mat}_\Lambda(1|1) \to \Lambda_0$ and Berezinian (superdeterminant) defined as $\mathrm{Ber} : \mathrm{Mat}_\Lambda(1|1) \backslash \{M | \, \mathfrak{r}_{body}(b) = 0\} \to \Lambda_0$ are

$$\mathrm{str} M = a - b, \tag{2}$$

[2]The supermatrix theory per se has own problems [1, 31, 30], unexpected conclusions [2, 55] and renewed standard theorems [59], which as a whole attach importance to more deep investigation of supermatrix systems from the abstract viewpoint.

$$\mathrm{Ber} M = \frac{a}{b} + \frac{\beta\alpha}{b^2}. \tag{3}$$

In [19] we introduced two kinds of possible reductions of M.

Definition 1 *Even-reduced supermatrices are elements from* $\mathrm{Mat}_\Lambda(1|1)$ *of the form*

$$M_{even} \equiv \begin{pmatrix} a & \alpha \\ 0 & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{even}(1|1) \subset \mathrm{Mat}_\Lambda(1|1). \tag{4}$$

Odd-reduced supermatrices are elements from $\mathrm{Mat}_\Lambda(1|1)$ *of the form*

$$M_{odd} \equiv \begin{pmatrix} 0 & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{odd}(1|1) \subset \mathrm{Mat}_\Lambda(1|1). \tag{5}$$

The odd-reduced supermatrices have a nilpotent Berezinian

$$\mathrm{Ber} M_{odd} = \frac{\beta\alpha}{b^2} \Rightarrow (\mathrm{Ber} M_{odd})^2 = 0 \tag{6}$$

and satisfy

$$M_{odd}^n = b^{n-2} \begin{pmatrix} \alpha\beta & \alpha b \\ \beta b & b^2 - (n-1)\alpha\beta \end{pmatrix}, \tag{7}$$

which gives $\mathrm{Ber} M_{odd}^n = 0$ and $\mathrm{str}\, M_{odd}^n = b^{n-2}(n\alpha\beta - b^2)$.

It is seen that $\mathbf{M}$ is a set sum of $\mathbf{M}_{even}$ and $\mathbf{M}_{odd}$

$$\mathbf{M} = \mathbf{M}_{even} \cup \mathbf{M}_{odd}. \tag{8}$$

The even- and odd-reduced supermatrices are mutually dual in the sense of the Berezinian addition formula [19]

$$\mathrm{Ber} M = \mathrm{Ber} M_{even} + \mathrm{Ber} M_{odd}. \tag{9}$$

The matrices from $\mathrm{Mat}(1|1)$ form a linear semigroup of $(1+1)\times(1+1)$-supermatrices under the standard supermatrix multiplication $\mathfrak{M}(1|1) \stackrel{def}{=} \{\mathbf{M} \mid \cdot\}$ [5]. Obviously, the even-reduced matrices $\mathbf{M}_{even}$ form a semigroup $\mathfrak{M}_{even}(1|1)$ which is a subsemigroup of $\mathfrak{M}(1|1)$, because of $\mathbf{M}_{even} \cdot \mathbf{M}_{even} \subseteq \mathbf{M}_{even}$.

In general the odd-reduced matrices M_{odd} do not form a semigroup, since their multiplication is not closed

$$M_{odd(1)}M_{odd(2)} = \begin{pmatrix} \alpha_1\beta_2 & \alpha_1 b_2 \\ b_1\beta_2 & b_1 b_2 + \beta_1\alpha_2 \end{pmatrix} \notin \mathbf{M}_{odd}. \tag{10}$$

Nevertheless, some subset of $\mathbf{M}_{odd}$ can form a semigroup. Indeed, due to the existence of zero divisors in Λ, from (10) it follows that

$$\mathbf{M}_{odd} \cdot \mathbf{M}_{odd} \cap \mathbf{M}_{odd} \neq \emptyset \quad \Rightarrow \quad \alpha\beta = 0. \tag{11}$$

Proposition 2 *1) The subsets $\mathbf{M}_{odd}|_{\alpha\beta=0} \subset \mathbf{M}_{odd}$ of the odd-reduced matrices satisfying $\alpha\beta = 0$ form a subsemigroup of $\mathfrak{M}(1|1)$ under the standard supermatrix multiplication.*

2) In this semigroup the subset of matrices with $\beta = 0$ is a left ideal, and one with $\alpha = 0$ is a right ideal, the matrices with $b = 0$ form a two-sided ideal.

Proof. Directly follows from (10). □

Definition 3 *An odd-reduced semigroup $\mathfrak{M}_{odd}(1|1)$ is a subsemigroup of $\mathfrak{M}(1|1)$ formed by the odd-reduced matrices $\mathbf{M}_{odd}$ satisfying $\alpha\beta = 0$.*

4 One parameter subsemigroups of odd-reduced semigroup

Let us investigate one-parameter subsemigroups of $\mathfrak{M}_{odd}(1|1)$. The simplest one is a semigroup of antidiagonal nilpotent supermatrices of the shape

$$Y_\alpha(t) \stackrel{def}{=} \begin{pmatrix} 0 & \alpha t \\ \alpha & 0 \end{pmatrix}. \tag{12}$$

Together with a null supermatrix

$$Z \stackrel{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{13}$$

they form a continuous null semigroup $\mathfrak{z}_\alpha(1|1)$ having the null multiplication

$$Y_\alpha(t)\,Y_\alpha(u) = Z \tag{14}$$

(cf. [11]).

Assertion 4 *For any fixed $t \in \Lambda^{1|0}$ the set $\{Y_\alpha(t), Z\}$ is a 0-minimal ideal in $\mathfrak{z}_\alpha(1|1)$.*

In search of nontrivial one parameter subsemigroups $\mathfrak{M}_{odd}(1|1)$ we consider the odd-reduced supermatrices of the following shape

$$P_\alpha(t) \stackrel{def}{=} \begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix} \tag{15}$$

where $t \in \Lambda^{1|0}$ is an even parameter of the Grassmann algebra Λ which "numbers" elements $P_\alpha(t)$ and $\alpha \in \Lambda^{0|1}$ is a fixed odd element of Λ which "numbers" the sets $\bigcup\limits_t P_\alpha(t)$.

Here we will study one-parameter subsemigroups in $\mathfrak{M}_{odd}(1|1)$ as abstract semigroups [11, 36], but not as semigroups of operators [13, 28], which will be done elsewhere.

First, we establish multiplication properties of $P_\alpha(t)$ supermatrices. From (10) and (15) it is seen that

$$\begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} 0 & \alpha u \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix} \tag{16}$$

Corollary 5 *The multiplication (16) is associative and so the $P_\alpha(t)$ supermatrices form a semigroup $\mathbf{P}_\alpha$.*

Corollary 6 *All $P_\alpha(t)$ supermatrices are idempotent*

$$\begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix}. \tag{17}$$

Proposition 7 *If $P_\alpha(t) = P_\alpha(u)$, then*

$$t - u = \operatorname{Ann}\alpha. \tag{18}$$

Proof. From the definition (15) it follows that two $P_\alpha(t)$ supermatrices are equal iff $\alpha t = \alpha u$, which gives (18). □

Similarly we can introduce idempotent $Q_\alpha(t)$ supermatrices of the shape

$$Q_\alpha(t) \stackrel{def}{=} \begin{pmatrix} 0 & \alpha \\ \alpha t & 1 \end{pmatrix} \tag{19}$$

which satisfy

$$\begin{pmatrix} 0 & \alpha \\ \alpha t & 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ \alpha u & 1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha u & 1 \end{pmatrix} \tag{20}$$

and form a semigroup $\mathbf{Q}_\alpha$.

Assertion 8 *The semigroups $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ contain no two sided zeros and identities.*

Assertion 9 *The semigroups $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ are continuous unions of one element groups with the action (17).*

The relations (16)–(20) and

$$\begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ \alpha u & 1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha t \\ \alpha u & 1 \end{pmatrix} \stackrel{def}{=} F_{tu}, \tag{21}$$

$$\begin{pmatrix} 0 & \alpha \\ \alpha u & 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 1 \end{pmatrix} \stackrel{def}{=} E \tag{22}$$

are important from the abstract viewpoint and will be exploited below.

Remark. In general the supermatrix multiplication is noncommutative, non-invertible, but associative, therefore any objects admitting supermatrix representation (with closed multiplication) are automatically semigroups.

5 Continuous supermatrix representation of zero semigroups

Let we consider an abstract set $\mathfrak{P}_\alpha$ which consists of elements $\mathbf{p}_t \in \mathfrak{P}_\alpha$ ($t \in \Lambda^{1|0}$ is a continuous parameter) satisfying the multiplication law

$$\mathbf{p}_t * \mathbf{p}_u = \mathbf{p}_t. \tag{23}$$

Assertion 10 *The multiplication (23) is associative and therefore the set $\mathfrak{P}_\alpha$ form a semigroup $\mathcal{P}_\alpha \overset{def}{=} \{\mathfrak{P}_\alpha; *\}$.*

Assertion 11 *The semigroup $\mathcal{P}_\alpha$ is isomorphic to the left zero semigroup [11] in which every element is both a left zero and a right identity.*

Proposition 12 *The semigroup $\mathcal{P}_\alpha$ is epimorphic to the semigroup $\mathbf{P}_\alpha$.*

Proof. Comparing (16) and (23) we observe that the mapping $\varphi : \mathcal{P}_\alpha \to \mathbf{P}_\alpha$ is a homomorphism. It is seen that two elements $\mathbf{p}_t$ and $\mathbf{p}_u$ satisfying (18) have the same image, i.e.

$$\varphi(\mathbf{p}_t) = \varphi(\mathbf{p}_u) \Leftrightarrow t - u = \operatorname{Ann}\alpha,\ \mathbf{p}_t, \mathbf{p}_u \in \mathfrak{P}_\alpha. \tag{24}$$

□

Definition 13 *The relation*

$$\Delta_\alpha = \{(\mathbf{p}_t, \mathbf{p}_u) \mid t - u = \operatorname{Ann}\alpha, \mathbf{p}_t, \mathbf{p}_u \in \mathfrak{P}_\alpha\}. \tag{25}$$

is called α-equality relation.

Remark. If the superparameter t and α take value in different Grassmann algebras which contain no mutually annihilating elements except zero, then $\operatorname{Ann}\alpha = 0$ and $\Delta_\alpha = \Delta$.

In the most of statements here the α-equality relation Δ_α substitutes formally the standard equality relation Δ, nevertheless the fact that $\Delta \neq \Delta_\alpha$ leads to some new structures and results. Among latter the following

Corollary 14 $\operatorname{Ker}\varphi = \bigcup_{t\in \operatorname{Ann}\alpha} \mathbf{p}_t$.

Remark. Outside $\operatorname{Ker}\varphi$ the semigroup $\mathcal{P}_\alpha$ is continuous and supersmooth, which can be shown by means of standard methods of superanalysis [5, 63].

Assertion 15 *The semigroup $\mathcal{P}_\alpha$ is not reductive and not cancellative, since $\mathbf{p} * \mathbf{p}_t = \mathbf{p} * \mathbf{p}_u \Rightarrow \mathbf{p}_t \Delta_\alpha \mathbf{p}_u$, but not $\mathbf{p}_t = \mathbf{p}_u$ for all $\mathbf{p} \in \mathfrak{P}_\alpha$. Therefore, the supermatrix representation given by φ is not faithful.*

Corollary 16 *If $t + \operatorname{Ann}\alpha \cap u + \operatorname{Ann}\alpha \neq \emptyset$, then $\mathbf{p}_t \Delta_\alpha \mathbf{p}_u$.*

Similarly the semigroup $\mathcal{Q}_\alpha$ with the multiplication

$$\mathbf{q}_t * \mathbf{q}_u = \mathbf{q}_u \tag{26}$$

is isomorphic to the right zero semigroup in which every element is both a right zero and a left identity and epimorphic to the semigroup $\mathbf{Q}_\alpha$.

Definition 17 *The semigroups $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ can be named "somewhere commutative"[3] or "almost anticommutative", since for them $\mathbf{p}_t * \mathbf{p}_u = \mathbf{p}_u * \mathbf{p}_t$ or $\mathbf{q}_t * \mathbf{q}_u = \mathbf{q}_u * \mathbf{q}_t$ gives $\alpha t = \alpha u$ and $t = u + \operatorname{Ann}\alpha$.*

Proposition 18 *The semigroups $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ are regular, but not inverse.*

Proof. For any two elements $\mathbf{p}_t$ and $\mathbf{p}_u$ using (23) we have $\mathbf{p}_t * \mathbf{p}_u * \mathbf{p}_t = (\mathbf{p}_t * \mathbf{p}_u) * \mathbf{p}_t = \mathbf{p}_t * \mathbf{p}_t = \mathbf{p}_t$. Similarly for $\mathbf{q}_t$ and $\mathbf{q}_u$. Then $\mathbf{p}_t$ has at least one inverse element $\mathbf{p}_u * \mathbf{p}_t * \mathbf{p}_u = \mathbf{p}_u$. But $\mathbf{p}_u$ is arbitrary, therefore in semigroups $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ any two elements are inverse. However, they are not inverse semigroups in which every element has a unique inverse [11]. □

The ideal structure of $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ differs somehow from the one of the left and right zero semigroups.

Proposition 19 *Each element from $\mathcal{P}_\alpha$ forms by itself a principal right ideal, each element from $\mathcal{Q}_\alpha$ forms a principal left ideal, and therefore every principal right and left ideals in $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ respectively have an idempotent generator.*

[3] By analogy with nowhere commutative rectangular bands [11].

Proof. From (23) and (26) it follows that $\mathbf{p}_t = \mathbf{p}_t * \mathfrak{P}_\alpha$ and $\mathbf{q}_u = \mathfrak{Q}_\alpha * \mathbf{q}_u$. □

Proposition 20 *The semigroups $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ are left and right simple respectively.*

Proof. It is seen from (23) and (26) that $\mathfrak{P}_\alpha = \mathfrak{P}_\alpha * \mathbf{p}_t$ and $\mathfrak{Q}_\alpha = \mathbf{q}_u * \mathfrak{Q}_\alpha$. □

The Green's relations on the standard left zero semigroup are the following: $\mathcal{L}$-equivalence coincides with the universal relation, and $\mathcal{R}$-equivalence coincides with the equality relation [11]. In our case the first statement is the same, but instead of the latter we have

Theorem 21 *In $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ respectively $\mathcal{R}$-equivalence and $\mathcal{L}$-equivalence coincide with the α-equality relation (25).*

Proof. Consider the $\mathcal{R}$-equivalence in $\mathcal{P}_\alpha$. The elements $\mathbf{p}_t, \mathbf{p}_u \in \mathcal{P}_\alpha$ are $\mathcal{R}$-equivalent iff $\mathbf{p}_t * \mathfrak{P}_\alpha = \mathbf{p}_u * \mathfrak{P}_\alpha$. Using (25) we obtain $\mathcal{R} = \Delta_\alpha$. □

6 Wreath rectangular band

Now we unify $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ semigroups in some nontrivial semigroup. First we consider the unified set of elements $\mathfrak{P}_\alpha \cup \mathfrak{Q}_\alpha$ and study their multiplication properties. Using (21) and (22) we notice that $\mathfrak{P}_\alpha \cap \mathfrak{Q}_\alpha = \mathbf{e}$, where $\varphi(\mathbf{e}) = E$ from (22), and therefore $\mathbf{e}\boldsymbol{\Delta}_\alpha \mathbf{p}_{t=1}$ and $\mathbf{e}\boldsymbol{\Delta}_\alpha \mathbf{q}_{t=1}$. So we are forced to distinguish the region $t = 1 + \operatorname{Ann}\alpha$ from other points in the parameter superspace $\Lambda^{1|0}$, and in what follows for any indices of $\mathbf{p}_t$ and $\mathbf{q}_t$ we imply $t \neq 1 + \operatorname{Ann}\alpha$.

Assertion 22 $\mathbf{e}$ *is the left zero and right identity for* $\mathbf{p}_t$*, and* $\mathbf{e}$ *is the right zero and left identity for* $\mathbf{q}_u$*, i.e.* $\mathbf{e} * \mathbf{p}_t = \mathbf{e}$, $\mathbf{p}_t * \mathbf{e} = \mathbf{p}_t$*, and* $\mathbf{q}_u * \mathbf{e} = \mathbf{e}$, $\mathbf{e} * \mathbf{q}_u = \mathbf{q}_u$.

Using (22) it is easily to check that $\mathbf{q}_u * \mathbf{p}_t = \mathbf{e}$, but the reverse product needs to consider additional elements which are not included in $\mathfrak{P}_\alpha \cup \mathfrak{Q}_\alpha$. From (21) we derive that

$$\mathbf{r}_{tu} = \mathbf{p}_t * \mathbf{q}_u, \tag{27}$$

where $\varphi(\mathbf{r}_{tu}) = F_{tu}$.

Let $\mathfrak{R}_\alpha \stackrel{def}{=} \bigcup_{t,u \neq 1 + \operatorname{Ann}\alpha} \mathbf{r}_{tu}$.

Definition 23 *A wreath rectangular band $\mathcal{W}_\alpha$ is a set of idempotent elements $\mathfrak{W}_\alpha = \mathfrak{P}_\alpha \cup \mathfrak{Q}_\alpha \cup \mathfrak{R}_\alpha$ with a $*$-product (23) and the following Cayley table*

$1 \setminus 2$	$\mathbf{e}$	$\mathbf{p}_t$	$\mathbf{p}_u$	$\mathbf{q}_t$	$\mathbf{q}_u$	$\mathbf{r}_{tu}$	$\mathbf{r}_{ut}$	$\mathbf{r}_{tw}$	$\mathbf{r}_{vw}$
$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{q}_t$	$\mathbf{q}_u$	$\mathbf{q}_u$	$\mathbf{q}_t$	$\mathbf{q}_w$	$\mathbf{q}_w$
$\mathbf{p}_t$	$\mathbf{p}_t$	$\mathbf{p}_t$	$\mathbf{p}_t$	$\mathbf{r}_{tt}$	$\mathbf{r}_{tu}$	$\mathbf{r}_{tu}$	$\mathbf{r}_{tt}$	$\mathbf{r}_{tw}$	$\mathbf{r}_{tw}$
$\mathbf{p}_u$	$\mathbf{p}_u$	$\mathbf{p}_u$	$\mathbf{p}_u$	$\mathbf{r}_{ut}$	$\mathbf{r}_{uu}$	$\mathbf{r}_{uu}$	$\mathbf{r}_{ut}$	$\mathbf{r}_{uw}$	$\mathbf{r}_{uw}$
$\mathbf{q}_t$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{q}_t$	$\mathbf{q}_u$	$\mathbf{q}_u$	$\mathbf{q}_t$	$\mathbf{q}_w$	$\mathbf{q}_w$
$\mathbf{q}_u$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{q}_t$	$\mathbf{q}_u$	$\mathbf{q}_u$	$\mathbf{q}_t$	$\mathbf{q}_w$	$\mathbf{q}_w$
$\mathbf{r}_{tu}$	$\mathbf{p}_t$	$\mathbf{p}_t$	$\mathbf{p}_t$	$\mathbf{r}_{tt}$	$\mathbf{r}_{tu}$	$\mathbf{r}_{tu}$	$\mathbf{r}_{tt}$	$\mathbf{r}_{tw}$	$\mathbf{r}_{tw}$
$\mathbf{r}_{ut}$	$\mathbf{p}_u$	$\mathbf{p}_u$	$\mathbf{p}_u$	$\mathbf{r}_{ut}$	$\mathbf{r}_{uu}$	$\mathbf{r}_{uu}$	$\mathbf{r}_{ut}$	$\mathbf{r}_{uw}$	$\mathbf{r}_{uw}$
$\mathbf{r}_{tw}$	$\mathbf{p}_t$	$\mathbf{p}_t$	$\mathbf{p}_t$	$\mathbf{r}_{tt}$	$\mathbf{r}_{tu}$	$\mathbf{r}_{tu}$	$\mathbf{r}_{tt}$	$\mathbf{r}_{tw}$	$\mathbf{r}_{tw}$
$\mathbf{r}_{vw}$	$\mathbf{p}_v$	$\mathbf{p}_v$	$\mathbf{p}_v$	$\mathbf{r}_{vt}$	$\mathbf{r}_{vu}$	$\mathbf{r}_{vu}$	$\mathbf{r}_{vt}$	$\mathbf{r}_{vw}$	$\mathbf{r}_{vw}$

which is associative[4] *(as it should be).*

From the Cayley table we can observe the following continuous subsemigroups in the wreath rectangular band:

- $\mathbf{e}$ - one element "near identity" subsemigroup;
- $\tilde{\mathcal{P}}_\alpha = \left\{ \bigcup_{t\neq 1+\operatorname{Ann}\alpha} \mathbf{p}_t; * \right\}$ – "reduced" left zero semigroup;
- $\mathcal{P}_\alpha = \left\{ \bigcup_{t\neq 1+\operatorname{Ann}\alpha} \mathbf{p}_t \cup \mathbf{e}; * \right\}$ – full left zero semigroup;
- $\tilde{\mathcal{Q}}_\alpha = \left\{ \bigcup_{t\neq 1+\operatorname{Ann}\alpha} \mathbf{q}_t; * \right\}$ – "reduced" right zero semigroup;
- $\mathcal{Q}_\alpha = \left\{ \bigcup_{t\neq 1+\operatorname{Ann}\alpha} \mathbf{q}_t \cup \mathbf{e}; * \right\}$ – full right zero semigroup;

[4]For convenience and clearness we display some additional relations.

- $\tilde{\mathcal{F}}_\alpha^{(1|1)} = \left\{ \bigcup_{t,u \neq 1+\mathrm{Ann}\,\alpha} \mathbf{r}_{tu}; * \right\}$ – "reduced" rectangular band;
- $\mathcal{F}_\alpha^{(1|1)} = \left\{ \bigcup_{t,u \neq 1+\mathrm{Ann}\,\alpha} \mathbf{r}_{tu} \cup \mathbf{e}; * \right\}$ – full rectangular band;
- $\mathcal{V}_\alpha^L = \left\{ \bigcup_{t,u \neq 1+\mathrm{Ann}\,\alpha} \mathbf{r}_{tu} \cup \mathbf{p}_t; * \right\}$ – "mixed" left rectangular band;
- $\mathcal{V}_\alpha^R = \left\{ \bigcup_{t,u \neq 1+\mathrm{Ann}\,\alpha} \mathbf{r}_{tu} \cup \mathbf{q}_t; * \right\}$ – "mixed" right rectangular band.

Thus we obtained the continuous supermatrix representation for the left and right zero semigroups and constructed from it the rectangular band supermatrix representation. It is well known that any rectangular band isomorphic to a Cartesian product of the left and right zero semigroups [29, 45]. Here we derived manifestly that (see (27)) and presented the concrete construction (21). In addition, we unified all the above semigroups in one object, viz. a wreath rectangular band.

7 Rectangular band continuous representation

The rectangular band multiplication is presented in the right lower corner of the Cayley table. Usually [11, 29] it is defined by one relation

$$\mathbf{r}_{tu} * \mathbf{r}_{vw} = \mathbf{r}_{tw}. \tag{28}$$

In our case the indices are even Grassmann parameters from $\Lambda^{1|0}$. As for zero semigroups that also leads to some special peculiarities in the ideal structure of such bands. Another difference is the absence of the condition $u = v$ which arises in some applications from the finite nature of indices considered as numbers of corresponding rows and columns in element matrices (see e.g. [32]).

Let us consider the Green's relations on $\mathcal{F}_\alpha^{(1|1)}$.

Proposition 24 *Any two elements in $\mathcal{F}_\alpha^{(1|1)}$ are $\mathcal{J}$- and $\mathcal{D}$-equivalent.*

Proof. From (28) we derive

$$\mathbf{r}_{tu} * \mathbf{r}_{vw} * \mathbf{r}_{tu} = \mathbf{r}_{tw} * \mathbf{r}_{tu} = \mathbf{r}_{tu}, \qquad (29)$$
$$\mathbf{r}_{vw} * \mathbf{r}_{tu} * \mathbf{r}_{vw} = \mathbf{r}_{vw} * \mathbf{r}_{tw} = \mathbf{r}_{vw}$$

for any $t, u, v, w \in \Lambda^{1|0}$. First we notice that these equalities coincide with the definition of $\mathcal{J}$-classes [11], therefore any two elements are $\mathcal{J}$-equivalent, and so $\mathcal{J}$ coincides with the universal relation on $\mathcal{F}_{\alpha}^{(1|1)}$. Next using (29) we observe that always $\mathbf{r}_{tu}\mathcal{R}\mathbf{r}_{tu} * \mathbf{r}_{vw}$ and $\mathbf{r}_{tu} * \mathbf{r}_{vw}\mathcal{L}\mathbf{r}_{vw}$. Since $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ (see e.g. [29]), then $\mathbf{r}_{tu}\mathcal{D}\mathbf{r}_{vw}$. □

Assertion 25 *Every $\mathcal{R}$-class $R_{\mathbf{r}_{tu}}$ consists of elements $\mathbf{r}_{tu}$ which are Δ_{α}-equivalent by the first index, i.e. $\mathbf{r}_{tu}\mathcal{R}\mathbf{r}_{vw} \Leftrightarrow t - v = \operatorname{Ann}\alpha$, and every $\mathcal{L}$-class $L_{\mathbf{r}_{tu}}$ consists of elements $\mathbf{r}_{tu}$ which are Δ_{α}-equivalent by the second index, i.e. $\mathbf{r}_{tu}\mathcal{L}\mathbf{r}_{vw} \Leftrightarrow u - w = \operatorname{Ann}\alpha$.*

Proof. That follows from (29), the manifest rectangular band decomposition (27) and Theorem 21. □

So that the intersection of $\mathcal{L}$- and $\mathcal{R}$-classes is nonempty. For the ordinary rectangular bands every $\mathcal{H}$-class consists of a single element [11, 29]. In our case the situation is more complicated.

Definition 26 *The relation*

$$\Delta_{\alpha}^{(1|1)} = \left\{ (\mathbf{r}_{tu}, \mathbf{r}_{vw}) \mid t - v = \operatorname{Ann}\alpha,\ u - w = \operatorname{Ann}\alpha,\ \mathbf{r}_{tu}, \mathbf{r}_{vw} \in \mathfrak{R}_{\alpha} \right\}. \qquad (30)$$

is called a double α-equality relation.

Theorem 27 *Every $\mathcal{H}$-class of $\mathcal{F}_{\alpha}^{(1|1)}$ consists of double $\Delta_{\alpha}^{(1|1)}$-equivalent elements satisfying $\mathbf{r}_{tu}\Delta_{\alpha}^{(2)}\mathbf{r}_{vw}$, and so $\mathcal{H} = \Delta_{\alpha}^{(1|1)}$.*

Proof. From (29) and the definitions (21) it follows that the intersection of $\mathcal{L}$- and $\mathcal{R}$-classes happens when $\alpha t = \alpha v$ and $\alpha u = \alpha w$. That gives $t = v + \operatorname{Ann}\alpha$, $u = w + \operatorname{Ann}\alpha$ which coincides with the double α-equality relation (30). □

Let us consider the mapping $\psi : \mathcal{F}_{\alpha}^{(1|1)} \to \mathcal{F}_{\alpha}^{(1|1)}/\mathcal{R} \times \mathcal{F}_{\alpha}^{(1|1)}/\mathcal{L}$ which maps an element $\mathbf{r}_{tu}$ to its $\mathcal{R}$- and $\mathcal{L}$-classes by

$$\psi\left(\mathbf{r}_{tu}\right) = \left\{ R_{\mathbf{r}_{tu}}, L_{\mathbf{r}_{tu}} \right\}. \qquad (31)$$

In the standard case ψ is a bijection [29]. Now we have

Assertion 28 *The mapping ψ is a surjection.*

Proof. That follows from Theorem 21 and the decomposition (27). □

Let the Cartesian product $\mathcal{F}_{\alpha}^{(1|1)}/\mathcal{R}\times\mathcal{F}_{\alpha}^{(1|1)}/\mathcal{L}$ is furnished with the rectangular band $\star$-multiplication of its $\mathcal{R}$- and $\mathcal{L}$-classes analogous to (28), i.e.

$$\{R_{\mathbf{r}_{tu}},L_{\mathbf{r}_{tu}}\}\star\{R_{\mathbf{r}_{vw}},L_{\mathbf{r}_{vw}}\}=\{R_{\mathbf{r}_{tu}},L_{\mathbf{r}_{vw}}\}\,. \tag{32}$$

For the standard rectangular bands the mapping ψ is an isomorphism [29, 41]. In our case we have

Theorem 29 *The mapping ψ is an epimorphism.*

Proof. First we observe from (29) that

$$\begin{aligned} R_{\mathbf{r}_{tu}*\mathbf{r}_{vw}} &= R_{\mathbf{r}_{tu}}, \\ L_{\mathbf{r}_{tu}*\mathbf{r}_{vw}} &= L_{\mathbf{r}_{vw}}, \end{aligned} \tag{33}$$

and so under the $\star$-multiplication (32) the mapping ψ is a homomorphism, since

$$\begin{aligned} \psi\left(\mathbf{r}_{tu}*\mathbf{r}_{vw}\right) &= \{R_{\mathbf{r}_{tu}*\mathbf{r}_{vw}},L_{\mathbf{r}_{tu}*\mathbf{r}_{vw}}\}=\{R_{\mathbf{r}_{tu}},L_{\mathbf{r}_{vw}}\} \\ &= \{R_{\mathbf{r}_{tu}},L_{\mathbf{r}_{tu}}\}\star\{R_{\mathbf{r}_{vw}},L_{\mathbf{r}_{vw}}\}=\psi\left(\mathbf{r}_{tu}\right)*\psi\left(\mathbf{r}_{vw}\right). \end{aligned} \tag{34}$$

Then a surjective homomorphism is an epimorphism (e.g. [26, 27]). □

8 Higher $(n|n)$-band continuous representations

Almost all above results can be generalized for the higher rectangular $(n|n)$ bands containing $2n$ continuous even Grassmann parameters. The corresponding matrix construction is

$$F_{t_1t_2\ldots t_n,u_1u_2\ldots u_n}\overset{def}{=}\begin{pmatrix} 0 & \alpha t_1\ \alpha t_2\ \ldots\ \alpha t_n \\ \alpha u_1 & \\ \alpha u_2 & I\left(n\times n\right) \\ \vdots & \\ \alpha u_n & \end{pmatrix}\in\mathrm{RMat}_{\Lambda}^{odd}\left(1|n\right), \tag{35}$$

where $t_1, t_2 \ldots t_n, u_1, u_2 \ldots u_n \in \Lambda^{1|0}$ are even parameters, $\alpha \in \Lambda^{1|0}$, $I\,(n \times n)$ is the unit matrix, and the matrix multiplication is

$$F_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n} F_{t'_1 t'_2 \ldots t'_n, u'_1 u'_2 \ldots u'_n} = F_{t_1 t_2 \ldots t_n, u'_1 u'_2 \ldots u'_n}. \tag{36}$$

Thus the idempotent supermatrices $F_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n}$ form a semigroup $\mathbf{F}_\alpha^{(n|n)}$.

Definition 30 *A higher $(n|n)$-band $\mathcal{F}_\alpha^{(n|n)} \ni \mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n}$ is represented by the supermatrices $\mathrm{RMat}_\Lambda^{odd}(1|n)$ of the form (35).*

The results of the Section 5 with some slight differences hold valid for $\mathcal{F}_\alpha^{(n|n)}$ as well.

Definition 31 *In $\mathcal{F}_\alpha^{(n|n)}$ the relation*

$$\begin{gathered}\Delta_\alpha^{(n|n)} \overset{def}{=} \{ \left(\mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n}, \mathbf{f}_{t'_1 t'_2 \ldots t'_n, u'_1 u'_2 \ldots u'_n} \right) \mid t_k - t'_k = \mathrm{Ann}\,\alpha, \\ u_k - u'_k = \mathrm{Ann}\,\alpha,\ 1 \leq k \leq n,\ \mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n}, \mathbf{f}_{t'_1 t'_2 \ldots t'_n, u'_1 u'_2 \ldots u'_n} \in \mathcal{F}_\alpha^{(2n)} \}\end{gathered} \tag{37}$$

is called a $(n|n)$-ple α-equality relation.

The semigroup $\mathcal{F}_\alpha^{(n|n)}$ is also epimorphic to $\mathbf{F}_\alpha$, and two $\Delta_\alpha^{(n|n)}$-equivalent elements of $\mathcal{F}_\alpha^{(n|n)}$ have the same image.

Let us consider $\mathrm{RMat}_\Lambda^{odd}(k|m)$ idempotent supermatrices of the shape

$$F_{TU} \overset{def}{=} \begin{pmatrix} 0 & \alpha T \\ \alpha U & I \end{pmatrix}, \tag{38}$$

where $T\,(k \times m)$ and $U\,(m \times k)$ are the band even parameter ordinary matrices and $I\,(m \times m)$ is the unit matrix. This band contains maximum $2km$ parameters from $\Lambda^{1|0}$.

The multiplication is

$$\begin{pmatrix} 0 & \alpha T \\ \alpha U & I \end{pmatrix} \begin{pmatrix} 0 & \alpha T' \\ \alpha U' & I \end{pmatrix} = \begin{pmatrix} 0 & \alpha T \\ \alpha U' & I \end{pmatrix}, \tag{39}$$

which coincides in block form with the rectangular band multiplication (28)

$$F_{TU} F_{T'U'} = F_{TU'}. \tag{40}$$

Theorem 32 *If $n = km$ the representations given by (35) and (38) are isomorphic.*

Proof. Since in (36) and (40) there exist no multiplication between parameters, and so the representations given by matrices (35) and (38) differ by permutation if $n = km$. $\square$

Corollary 33 *The supermatrices* $\mathrm{RMat}_{\Lambda}^{odd}(1|n)$ *of the shape (35) exhaust all possible* $(n|n)$*-band continuous representations.*

Remark. The supermatrices (35) represent $(k|m)$-bands as well, where $1 \leq k \leq n$, $1 \leq m \leq n$. In this situation $t_{k+1} = 1 + \mathrm{Ann}\,\alpha, \ldots t_n = 1 + \mathrm{Ann}\,\alpha$, $u_{m+1} = 1+\mathrm{Ann}\,\alpha, \ldots u_n = 1+\mathrm{Ann}\,\alpha$. So the above isomorphism takes place for different bands having the same number of parameters Therefore, we will consider below mostly the full $(n|n)$-bands, implying that they contain all particular and reduced cases.

Remark. For $k = 0$ and $m = 0$ they describe m-right zero semigroups $\mathcal{Q}_{\alpha}^{(m)}$ and k-left zero semigroups $\mathcal{P}_{\alpha}^{(k)}$ respectively having the following multiplication laws (cf. (23) and (26))

$$\begin{aligned} \mathbf{q}_{u_1 u_2 \ldots u_m} * \mathbf{q}_{u'_1 u'_2 \ldots u'_m} &= \mathbf{q}_{u'_1 u'_2 \ldots u'_m}, \\ \mathbf{p}_{t_1 t_2 \ldots t_k} * \mathbf{p}_{t'_1 t'_2 \ldots t'_k} &= \mathbf{p}_{t_1 t_2 \ldots t_k}. \end{aligned} \tag{41}$$

Proposition 34 *The m-right zero semigroups $\mathcal{Q}_{\alpha}^{(m)}$ and k-left zero semigroups $\mathcal{P}_{\alpha}^{(k)}$ are irreducible in the sense that they cannot be presented as a direct product of "1-dimensional" right zero $\mathcal{Q}_{\alpha}$ and left zero $\mathcal{P}_{\alpha}$ semigroups respectively.*

Proof. It follows directly from comparing of the structure of supermatrices (15), (19) and (35). $\square$

Proposition 35 *For the purpose of constructing $(k|m)$-bands one cannot use "1-dimensional" right zero $\mathcal{Q}_{\alpha}$ and left zero $\mathcal{P}_{\alpha}$ semigroups, because they reduce it to the ordinary "2-dimensional" rectangular band.*

Proof. Indeed let $\tilde{\mathbf{f}}_{t_1t_2\ldots t_k,u_1u_2\ldots u_m} = \mathbf{p}_{t_1} * \mathbf{p}_{t_2} \ldots * \mathbf{p}_{t_k} * \mathbf{q}_{u_1} * \mathbf{q}_{u_2} \ldots * \mathbf{q}_{u_m}$. Then using the Cayley table above we derive $\tilde{\mathbf{f}}_{t_1t_2\ldots t_k,u_1u_2\ldots u_m} = \mathbf{p}_{t_1} * \mathbf{q}_{u_m}$ which trivially coincides with (27). Thus any combination of elements from "1-dimensional" right zero and left zero semigroups will not lead to new construction other than in the Cayley table. □

Instead we have the following decomposition of a $(k|m)$-band into k-left zero semigroup $\mathcal{P}_\alpha^{(k)}$ and m-right zero semigroups $\mathcal{Q}_\alpha^{(m)}$ k-left zero semigroup $\mathcal{P}_\alpha^{(k)}$

$$\mathbf{f}_{t_1t_2\ldots t_k,u'_1u'_2\ldots u'_m} = \mathbf{p}_{t_1t_2\ldots t_k} * \mathbf{q}_{u'_1u'_2\ldots u'_m}. \tag{42}$$

Despite this formula is similar to (27), we stress that the increasing of number of superparameters is not an artificial trick, but a natural way of searching for new constructions leading to generalization of Green's relations and fine ideal structure of $(n|n)$-bands, which has no analogs in the standard approach [11, 29, 45].

9 Fine ideal structure of $(n|n)$-bands

Let us consider the Green's relations for $(n|n)$-bands. We will try to establish the supermatrix meaning of properties of $\mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{H}$-classes. It will allow us to define and study new equivalences most naturally, as well as to clear the previous constructions. For clarity we use $(2|2)$-band representation, and the extending all the results to $(n|n)$-bands can be easily done without further detail explanations.

The exact shape of the $(2|2)$-band $\mathcal{F}_\alpha^{(2|2)} \ni \mathbf{f}_{t_1t_2,u_1u_2}$ supermatrix representation is

$$F_{t_1t_2,u_1u_2} = \begin{pmatrix} 0 & \alpha t_1 & \alpha t_2 \\ \alpha u_1 & 1 & 0 \\ \alpha u_2 & 0 & 1 \end{pmatrix}. \tag{43}$$

According to the definition of $\mathcal{R}$-classes [11], two elements $F_{t_1t_2,u_1u_2}$ and $F_{t'_1t'_2,u'_1u'_2}$ are $\mathcal{R}$-equivalent iff there exist two another elements $X_{x_1x_2,y_1y_2}$, $W_{v_1v_2,w_1w_2}$ such that $F_{t_1t_2,u_1u_2}X_{x_1x_2,y_1y_2} = F_{t'_1t'_2,u'_1u'_2}$ and $F_{t'_1t'_2,u'_1u'_2}W_{v_1v_2,w_1w_2} = F_{t_1t_2,u_1u_2}$ simultaneously. In manifest form

$$F_{t_1t_2,u_1u_2}X_{x_1x_2,y_1y_2} = \begin{pmatrix} 0 & \alpha t_1 & \alpha t_2 \\ \alpha y_1 & 1 & 0 \\ \alpha y_2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha t'_1 & \alpha t'_2 \\ \alpha u'_1 & 1 & 0 \\ \alpha u'_2 & 0 & 1 \end{pmatrix} \tag{44}$$

and

$$F_{t'_1t'_2,u'_1u'_2}W_{v_1v_2,w_1w_2} = \begin{pmatrix} 0 & \alpha t'_1 & \alpha t'_2 \\ \alpha w_1 & 1 & 0 \\ \alpha w_2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha t_1 & \alpha t_2 \\ \alpha u_1 & 1 & 0 \\ \alpha u_2 & 0 & 1 \end{pmatrix}. \tag{45}$$

To satisfy the last equalities in (44) and (45) we should choose

$$\begin{aligned} \alpha y_1 = \alpha u'_1,\ \alpha y_1 = \alpha u'_1, \\ \alpha w_1 = \alpha u_1,\ \alpha w_2 = \alpha u_2, \end{aligned} \tag{46}$$

and

$$\alpha t_1 = \alpha t'_1,\ \alpha t_2 = \alpha t'_2. \tag{47}$$

Due to the arbitrariness of $X_{x_1x_2,y_1y_2}$ and $W_{v_1v_2,w_1w_2}$ the first equalities (46) can be always solved by parameter choice. The second equations (47) are the definition of $\mathcal{R}$-class of (2|2)-band in the supermatrix interpretation. Thus we have the following general

Definition 36 *The $\mathcal{R}$-classes of $(n|n)$-band consist of elements having all (!) αt_k fixed, where $1 \leq k \leq n$.*

As the dual counterpart we formulate

Definition 37 *The $\mathcal{L}$-classes of $(n|n)$-band consist of elements having all (!) αu_k fixed, where $1 \leq k \leq n$.*

In such a picture it is obvious that the join of these relations $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$ covers all possible elements, and therefore any two elements in $(n|n)$-band are $\mathcal{D}$-equivalent (cf. Proposition 24) . The intersection of them $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ obviously consists of the elements with all (!) αt_k and αu_k fixed. Indeed there is here the source of the $(n|n)$-ple α-equality relation definition (37).

Proposition 38 *In (2|2)-band $\mathcal{J}$-relation coincides with the universal relation.*

Proof. Multiplying (44) by $F_{t_1t_2,u_1u_2}$ from the right and by $X_{x_1x_2,y_1y_2}$ from the left we obtain

$$\begin{aligned} F_{t_1t_2,u_1u_2}X_{x_1x_2,y_1y_2}F_{t_1t_2,u_1u_2} = F_{t_1t_2,u_1u_2}, \\ X_{x_1x_2,y_1y_2}F_{t_1t_2,u_1u_2}X_{x_1x_2,y_1y_2} = X_{x_1x_2,y_1y_2} \end{aligned} \tag{48}$$

for any $t_1, t_2, u_1, u_2, x_1, x_2, y_1, y_2 \in \Lambda^{(1|0)}$, which coincides with the definition of $\mathcal{J}$-relation. The arbitrariness of $F_{t_1 t_2, u_1 u_2}$ and $X_{x_1 x_2, y_1 y_2}$ proves the statement. $\square$

Summing up the standard approach for (2|2)-bands we have

$$\mathbf{f}_{t_1 t_2, u_1 u_2} \mathcal{R} \mathbf{f}_{t_1' t_2', u_1' u_2'} \Leftrightarrow \{\alpha t_1 = \alpha t_1' \wedge \alpha t_2 = \alpha t_2'\}, \tag{49}$$

$$\mathbf{f}_{t_1 t_2, u_1 u_2} \mathcal{L} \mathbf{f}_{t_1' t_2', u_1' u_2'} \Leftrightarrow \{\alpha u_1 = \alpha u_1' \wedge \alpha u_2 = \alpha u_2'\}, \tag{50}$$

$$\mathbf{f}_{t_1 t_2, u_1 u_2} \mathcal{D} \mathbf{f}_{t_1' t_2', u_1' u_2'} \Leftrightarrow \left\{ \begin{array}{c} (\alpha t_1 = \alpha t_1' \wedge \alpha t_2 = \alpha t_2') \vee \\ (\alpha u_1 = \alpha u_1' \wedge \alpha u_2 = \alpha u_2') \end{array} \right\}, \tag{51}$$

$$\mathbf{f}_{t_1 t_2, u_1 u_2} \mathcal{H} \mathbf{f}_{t_1' t_2', u_1' u_2'} \Leftrightarrow \left\{ \begin{array}{c} (\alpha t_1 = \alpha t_1' \wedge \alpha t_2 = \alpha t_2') \wedge \\ (\alpha u_1 = \alpha u_1' \wedge \alpha u_2 = \alpha u_2') \end{array} \right\}. \tag{52}$$

Now we are ready to introduce the fine ideal structure and understand what was missed by the standard approach. From (49) and (50) it is seen that the separate four possibilities for the equations to satisfy are not covered by the ordinary $\mathcal{R}$- and $\mathcal{L}$-equivalent relations. It is clear , why we wrote above exclamation marks: these statements will be revised. So we are forced to define more general relations, we call them "fine equivalent relations". They are appropriate to describe all possible classes of elements in $(n|n)$-bands missed by the standard approach. First we define them as applied to our particular case for clarity.

Definition 39 *The fine $\mathcal{R}^{(k)}$- and $\mathcal{L}^{(k)}$-relations on the* (2|2)*-band are defined by*

$$\mathbf{f}_{t_1 t_2, u_1 u_2} \mathcal{R}^{(1)} \mathbf{f}_{t_1' t_2', u_1' u_2'} \Leftrightarrow \{\alpha t_1 = \alpha t_1'\}, \tag{53}$$

$$\mathbf{f}_{t_1 t_2, u_1 u_2} \mathcal{R}^{(2)} \mathbf{f}_{t_1' t_2', u_1' u_2'} \Leftrightarrow \{\alpha t_2 = \alpha t_2'\}, \tag{54}$$

$$\mathbf{f}_{t_1 t_2, u_1 u_2} \mathcal{L}^{(1)} \mathbf{f}_{t_1' t_2', u_1' u_2'} \Leftrightarrow \{\alpha u_1 = \alpha u_1'\}, \tag{55}$$

$$\mathbf{f}_{t_1 t_2, u_1 u_2} \mathcal{L}^{(2)} \mathbf{f}_{t_1' t_2', u_1' u_2'} \Leftrightarrow \{\alpha u_2 = \alpha u_2'\}. \tag{56}$$

Proposition 40 *The fine $\mathcal{R}^{(k)}$- and $\mathcal{L}^{(k)}$-relations are equivalence relations.*

Proof. Follows from the manifest form of the multiplication and (44) and (45). □

Therefore, they divide $\mathcal{F}_\alpha^{(2|2)}$ to four fine equivalence classes $\mathcal{F}_\alpha^{(2|2)}/\mathcal{R}^{(k)}$ and $\mathcal{F}_\alpha^{(2|2)}/\mathcal{L}^{(k)}$ as follows

$$R_{\mathbf{f}}^{(1)} = \left\{ \mathbf{f}_{t_1 t_2, u_1 u_2} \in \mathcal{F}_\alpha^{(2|2)} \mid \alpha t_1 = const \right\}, \tag{57}$$

$$R_{\mathbf{f}}^{(2)} = \left\{ \mathbf{f}_{t_1 t_2, u_1 u_2} \in \mathcal{F}_\alpha^{(2|2)} \mid \alpha t_2 = const \right\}, \tag{58}$$

$$L_{\mathbf{f}}^{(1)} = \left\{ \mathbf{f}_{t_1 t_2, u_1 u_2} \in \mathcal{F}_\alpha^{(2|2)} \mid \alpha u_1 = const \right\}, \tag{59}$$

$$L_{\mathbf{f}}^{(2)} = \left\{ \mathbf{f}_{t_1 t_2, u_1 u_2} \in \mathcal{F}_\alpha^{(2|2)} \mid \alpha u_2 = const \right\}. \tag{60}$$

For clearness we can present schematically

$$\begin{array}{cc} & \begin{array}{ccc} \quad & R_{\mathbf{f}}^{(1)} & R_{\mathbf{f}}^{(2)} \\ & \updownarrow & \updownarrow \end{array} \\ \begin{array}{c} \\ L_{\mathbf{f}}^{(1)} \leftrightarrow \\ L_{\mathbf{f}}^{(2)} \leftrightarrow \end{array} & \begin{pmatrix} 0 & \alpha t_1 & \alpha t_2 \\ \alpha u_1 & 1 & 0 \\ \alpha u_2 & 0 & 1 \end{pmatrix} \end{array}, \tag{61}$$

where arrows show which element of the supermatrix is fixed according to a given fine equivalence relation.

From them we can obtain all known relations

$$\mathcal{R}^{(1)} \cap \mathcal{R}^{(2)} = \mathcal{R}, \tag{62}$$

$$\mathcal{L}^{(1)} \cap \mathcal{L}^{(2)} = \mathcal{L}, \tag{63}$$

and

$$\left(\mathcal{R}^{(1)} \cap \mathcal{R}^{(2)} \right) \cap \left(\mathcal{L}^{(1)} \cap \mathcal{L}^{(2)} \right) = \mathcal{H}, \tag{64}$$

$$\left(\mathcal{R}^{(1)} \cap \mathcal{R}^{(2)} \right) \vee \left(\mathcal{L}^{(1)} \cap \mathcal{L}^{(2)} \right) = \mathcal{D}. \tag{65}$$

However there are many other possible "mixed" equivalences which can be classified using the definitions

$$\mathcal{H}^{(i|j)} = \mathcal{R}^{(i)} \cap \mathcal{L}^{(j)}, \tag{66}$$

$$\mathcal{D}^{(i|j)} = \mathcal{R}^{(i)} \vee \mathcal{L}^{(j)}, \tag{67}$$

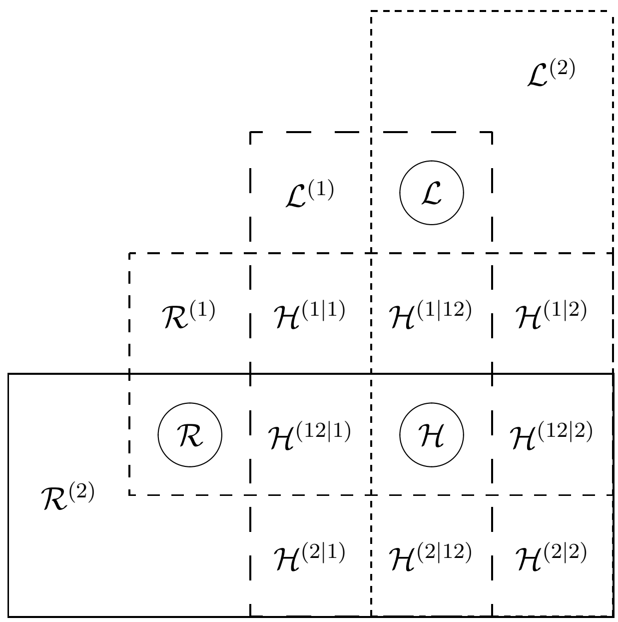
$$\mathcal{H}^{(ij|k)} = \left(\mathcal{R}^{(i)} \cap \mathcal{R}^{(j)}\right) \cap \mathcal{L}^{(k)}, \tag{68}$$

$$\mathcal{H}^{(i|kl)} = \mathcal{R}^{(i)} \cap \left(\mathcal{L}^{(k)} \cap \mathcal{L}^{(l)}\right), \tag{69}$$

$$\mathcal{D}^{(ij|k)} = \left(\mathcal{R}^{(i)} \cap \mathcal{R}^{(j)}\right) \vee \mathcal{L}^{(k)}, \tag{70}$$

$$\mathcal{D}^{(i|kl)} = \mathcal{R}^{(i)} \vee \left(\mathcal{L}^{(k)} \cap \mathcal{L}^{(l)}\right). \tag{71}$$

The graphic interpretation of the mixed equivalence relations is given by the following diagram



(72)

where the standard Green's relations are marked with circles. In (72) the standard $\mathcal{R}$- and $\mathcal{L}$-relations occupy 4 small squares longwise, the $\mathcal{H}^{(i|j)}$-relations occupy 4 small squares in square, the $\mathcal{H}^{(ij|k)}$- and $\mathcal{H}^{(i|jk)}$-relations occupy 2 small squares, the standard $\mathcal{H}$-relation occupies 1 small square.

We observe that the mixed relations (66)-(71) are "wider" in some sense than the standard ones (62)-(65). Therefore, using them we are able to describe thoroughly and appropriately all classes of elements from $(n|n)$-bands including those which are missed when one uses the standard Green's relations only[5].

[5]For nonnegative ordinary matrices the generalized Green's relations (in some different sense) were studied in [64].

For every mixed relation above we can determine a corresponding class using obvious definitions. Then for every mixed $\mathcal{D}$-class we can build the mixed eggbox diagram [11] of the fine $\mathcal{R},\mathcal{L}$-classes which will have so many dimensions how many terms a given mixed relation has in its right hand side of (67), (70) and (71). For instance, the eggbox diagrams of $\mathcal{D}^{(i|j)}$-classes are two dimensional, but ones of $\mathcal{D}^{(ij|k)}$ and $\mathcal{D}^{(i|jk)}$-classes should be 3-dimensional. In case of $(n|n)$-bands one has to consider all possible k-dimensional eggbox diagrams, where $2 \leq k \leq n-1$.

The introduced fine equivalence relations (53)-(56) admit a subsemigroup interpretation.

Lemma 41 *The elements of $\mathcal{F}_{\alpha}^{(n|n)}$ having $\alpha t_k = \beta_k$ and $\alpha u_k = \gamma_k$, where $\beta_k, \gamma_k \in \Lambda^{0|1}$ are fixed, and $1 \leq k \leq m$, form various m-index subsemigroups.*

Proof. It follows from the manifest form of matrix multiplication in (35). □

We consider $(n-1)$-index subsemigroups of $\mathcal{F}_{\alpha}^{(n|n)}$. They consist of elements having all but one αt_k and all but one αu_k fixed. Let

$$\mathcal{U}_{\alpha}^{(k)} \stackrel{def}{=} \left\{ \mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n} \in \mathcal{F}_{\alpha}^{(n|n)} \mid \bigwedge_{i \neq k} \alpha t_i = \beta_i \bigwedge_{i \neq k} \alpha u_i = \gamma_i \right\} \tag{73}$$

be a $(n-1)$-index subsemigroup which has only one nonfixed pair αt_k, αu_k. The Green's relations on the subsemigroup $\mathcal{U}_{\alpha}^{(k)}$ are the following

$$\mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n} \mathcal{R}_U^{(k)} \mathbf{f}_{t'_1 t'_2 \ldots t'_n, u'_1 u'_2 \ldots u'_n} \Leftrightarrow \{\alpha t_k = \alpha t'_k\}, \tag{74}$$

$$\mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n} \mathcal{L}_U^{(k)} \mathbf{f}_{t'_1 t'_2 \ldots t'_n, u'_1 u'_2 \ldots u'_n} \Leftrightarrow \{\alpha u_k = \alpha u'_k\}, \tag{75}$$

$$\mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n} \mathcal{H}_U^{(k)} \mathbf{f}_{t'_1 t'_2 \ldots t'_n, u'_1 u'_2 \ldots u'_n} \Leftrightarrow \{\alpha t_k = \alpha t'_k \wedge \alpha u_k = \alpha u'_k\}, \tag{76}$$

$$\mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n} \mathcal{D}_U^{(k)} \mathbf{f}_{t'_1 t'_2 \ldots t'_n, u'_1 u'_2 \ldots u'_n} \Leftrightarrow \{\alpha t_k = \alpha t'_k \vee \alpha u_k = \alpha u'_k\}, \tag{77}$$

where $\mathbf{f}_{t_1 t_2 \ldots t_n, u_1 u_2 \ldots u_n}, \mathbf{f}_{t'_1 t'_2 \ldots t'_n, u'_1 u'_2 \ldots u'_n} \in \mathcal{U}_{\alpha}^{(k)} \subset \mathcal{F}_{\alpha}^{(n|n)}$.

Theorem 42 *The Green's relations on $\mathcal{U}_{\alpha}^{(k)}$ are the restrictions of the corresponding fine relations (53)-(56) on $\mathcal{F}_{\alpha}^{(n|n)}$ to the subsemigroup $\mathcal{U}_{\alpha}^{(k)}$*

$$\mathcal{R}_U^{(k)} = \mathcal{R}^{(k)} \cap \left(\mathcal{U}_{\alpha}^{(k)} \times \mathcal{U}_{\alpha}^{(k)} \right), \tag{78}$$

$$\mathcal{L}_U^{(k)} = \mathcal{L}^{(k)} \cap \left(\mathcal{U}_\alpha^{(k)} \times \mathcal{U}_\alpha^{(k)} \right), \tag{79}$$

$$\mathcal{H}_U^{(k)} = \mathcal{H}^{(k|k)} \cap \left(\mathcal{U}_\alpha^{(k)} \times \mathcal{U}_\alpha^{(k)} \right), \tag{80}$$

$$\mathcal{D}_U^{(k)} = \mathcal{D}^{(k|k)} \cap \left(\mathcal{U}_\alpha^{(k)} \times \mathcal{U}_\alpha^{(k)} \right). \tag{81}$$

Proof. It is sufficient to prove the statement for the particular case of $\mathcal{F}_\alpha^{(2|2)}$ and $\mathcal{U}_\alpha^{(1)}$, and then to derive the general one by induction. Using the manifest form of $\mathcal{R}$-class definition (44)-(45) we conclude that the condition $\alpha t_1 = \alpha t_1'$ is common for the fine $\mathcal{R}^{(k)}$-classes and for the subsemigroup $\mathcal{R}_U^{(k)}$-classes. By analogy one can prove other equalities. □

Remark. The second condition $\alpha t_2 = \alpha t_2'$ (which is a second part of the definition of the ordinary $\mathcal{R}$-relation for $\mathcal{F}_\alpha^{(2|2)}$ (49)) holds in $\mathcal{U}_\alpha^{(1)}$ as well, but due to the subsemigroup own definition ($\alpha t_2 = \beta_2 = const$, $\alpha u_2 = \gamma_2 = const$), however $\alpha t_2 = \alpha t_2'$ does not enter to the fine relation $\mathcal{R}^{(k)}$ at all. Therefore the latter is the most general one among the $\mathcal{R}$-relations under consideration.

Remark. The Theorem 42 can be considered in view of [24], where the formulas similar to (78)-(80) were proved, but with ordinary Green's relations on the right hand side. Referring to the Diagram 72 we conclude that our result contains the ordinary case [24] as a particular one.

Moreover, we assume that the Theorem 42 has more deep sense and gives another treatment to the fine equivalence relations.

Conjecture 43 *The Green's relations on a subsemigroup U of S have as counterpart prolonged images in S indeed the fine equivalence relations on S.*

We proved this statement for the particular case of continuous $(n|n)$-bands. It would be interesting to find and investigate other possible algebraic systems where the Conjecture 43 is true.

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